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Propagation of light-pulses at a negative group-velocity

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Abstract. This paper deals with the apparent superluminal propagation of electromagnetic pulses in a linear dispersive medium. One specifically examines the possibility that the pulse leaving the medium may be nearly identical to the incident one (low distortion) and in significant advance of it (strongly negative group-delays). Favourable conditions are obtained in a dilute medium where the required anomalous dispersion originates in an ensemble of narrow absorption or gain lines. Analytical expressions of the advancement of the pulse centre-of-gravity and of the lowest order distortion are established from the transfer-function of the medium. The experiments already achieved with arrangements involving a single absorption-line or a gain-doublet are analysed in detail and compared. The considerable difficulties to overcome in order to attain advancements comparable to the pulse width without important distortion are pointed out. New and promising schemes involving a narrow dip in a gain profile or absorption-doublets are proposed.

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1 Introduction

It is well-known that the group velocity of an electromagnetic pulse can be negative in regions of anomalous dispersion [1]. For a long time, due to an incorrect reference to the causality principle and the theory of special relativity, it has been considered that such velocities have no physical significance and this analysis can still be found in classical textbooks [2]. Indeed it is clear that any information cannot be conveyed at a negative velocity (it would be received before having been emitted). This remark also applies to any localised feature in an electromagnetic pulse. One cannot however exclude the possibility that ideally smooth pulses may propagate in conditions such that the profile of the output pulse be nearly identical to that of the incident pulse and in advance of it. The existence of the phenomenon has been predicted more than 30 years ago by Garrett and McCumber [3] in the particular case of Gaussian pulses propagating in a resonant absorber and it has been experimentally evidenced in various conditions [4–9]. Its explanation lies in the fact that a given point of the output profile is not a direct reflection of the homologous point of the input profile but results from the action of the medium on all the earlier part of the pulse. From this viewpoint, it is irrelevant to make a direct correspondence between the maximums of the input and output pulses and the widespread statement that *the peak of the pulse exits the medium before it even enters it* may be confusing. As a matter of fact the superluminal propagation only concerns the overall profile of the pulse. The phenomenon is intriguing but not abnormal. In a spectral instead of temporal description, it exclusively results from interference between the different frequency components of the pulse in an anomalous dispersion region, as this occurs in a normal dispersion region.

Strongly negative group-delays, that is pulse advancements large with respect to the luminal propagation time, are easily obtained in media with narrow absorption or gain lines [4–10]. The experimental arrangements used up to now or in progress involve either a single absorption-line [4–6] or a gain doublet [7–9]. The challenge is to attain advancements comparable to the pulse-width with a low level of pulse-distortion. Our aim is here to study this problem in a frame as general as possible. In Section 2, we give the expression of the transfer function of a medium whose dispersion originates from an ensemble of narrow absorption or gain lines. General properties of the pulse advancement and of the lowest-order pulse-distortion are derived from this expression. In Sections 3 and 4, the previous results are applied to the analysis of the experiments already achieved or in progress with the single absorption-line and gain-doublet arrangements. New and promising schemes involving a narrow dip in a gain profile or absorption-doublets are proposed in Section 5. We conclude in Section 6 by summarising the main

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results of our work and stressing the severe constraints to the observation of significant pulse advancements.

2 General analysis

To be definite, we consider an electromagnetic pulse propagating in the z-direction in a medium of thickness L, with an electric field polarised according to the x-direction. The x-component of the field is written

$$E_x(z,t') = \mathrm{Re}[E(z,t')\exp(\mathrm{i}\omega_0 t')] \tag{1}$$

where t' is the retarded time ($t' = t - z/c$ in a dilute medium), ω_0 is a reference frequency[1] equal or nearly equal to the mean frequency of the incident pulse (hereafter denoted the working frequency) and $E(z,t')$ is the complex envelope of the pulse. Equation (1) implicitly assumes that the envelope $E(z,t')$ slowly varies at the scale of $1/\omega_0$ in time and of c/ω_0 in distance [11] or, equivalently, that its Fourier spectrum $\widehat{E}(z,\Omega)$ is concentrated in a narrow region close to the zero frequency ($|\Omega| \ll \omega_0$). From the viewpoint of the linear systems theory [12,13], the action of the medium is entirely characterised by the impulse response $h(t')$ linking the envelopes of the input and output pulses or by the transfer function $H(\Omega)$ linking their Fourier transforms

$$E(L,t') = h(t') \otimes E(0,t') \tag{2}$$

$$\widehat{E}(L,\Omega) = H(\Omega)\,\widehat{E}(0,\Omega) \tag{3}$$

with

$$h(t') = \int_{-\infty}^{+\infty} H(\Omega)\exp(\mathrm{i}\Omega t')\frac{\mathrm{d}\Omega}{2\pi}\cdot \tag{4}$$

Quite generally the transfer function may be written

$$H(\Omega) = \exp\left[F(\Omega) + \mathrm{i}\varphi(\Omega)\right] = \exp\left[\Gamma(\Omega)\right] = \exp\left[\mathrm{i}\Phi(\Omega)\right] \tag{5}$$

where $F(\Omega)$ is the (real) amplitude gain-factor, $\varphi(\Omega)$ the (real) phase-advancement, $\Gamma(\Omega)$ the complex gain-factor and $\Phi(\Omega)$ the complex phase. $|H(\Omega)| = \exp[F(\Omega)]$ is obviously the amplitude gain. For a medium whose dispersion originates from an ensemble of m homogeneously broadened lines, the complex gain-factor simply reads

$$\Gamma(\Omega) = \sum_{q=1}^{m} \frac{g_q L/2}{1 + \mathrm{i}\Omega/\gamma_q + \mathrm{i}\delta_q/\gamma_q} \tag{6}$$

with $\delta_q = (\omega_0 - \omega_q)$ and where ω_q, g_q and γ_q are respectively the frequency, the relaxation rate and the intensity gain coefficient of the qth line. For an absorption line, $g_q = -\alpha_q$, where α_q is the intensity absorption coefficient. Equation (6) is obtained by assuming that all the δ_q and γ_q are small with respect to ω_0 and the slowly varying envelope approximation requires that $\sum_{q=1}^{m} |g_q| \ll \omega_0/c$.

[1] All the frequencies introduced in this paper are angular frequencies.

All the previous assumptions and approximations are well verified in every realistic experiment.

All the poles of $\Gamma(p)$, obtained by substituting the complex variable p to $\mathrm{i}\Omega$ in $\Gamma(\Omega)$, have a negative real part and it is the same for $H(p)$ and $1/H(p)$. Otherwise said, $\Gamma(p)$ is the transfer function of a causal system and the medium is a minimum phase system [12,13]. The impulse response $h(t')$ of the medium is thus strictly zero for $t' < 0$ and exactly starts at the (retarded) time $t' = 0$. This means that any singularity in the incident pulse will generate a transient (wiggle) whose front will propagate at the luminal velocity, in agreement with the forerunners theory of Sommerfeld and Brillouin [1]. More precisely, the short-term behaviour of the impulse response is associated to the high frequency components of the transfer function ($\Omega \to \pm\infty$) and, expanding $H(\Omega)$ at the first order in $1/\Omega$, we easily get

$$h(t') \approx \delta(t') + U(t')\sum_{q=1}^{m}\frac{g_q L \gamma_q}{2} \tag{7}$$

where $\delta(t')$ and $U(t')$ are respectively the Dirac and unit step functions. Equation (7) obviously supports the general analysis and, besides, shows that a first order discontinuity will propagate while conserving its initial amplitude.

Two properties, which are not specific of the particular form of the transfer function, deserve a mention. The first one is quite general and is related to the (complex) area of the pulse-envelope [14]. By means of equation (3), the envelope of the output pulse may be written

$$\int_{-\infty}^{+\infty} E(L,t')\,\mathrm{d}t' = \widehat{E}(L,0) = H(0)\widehat{E}(0,0) = H(0)\int_{-\infty}^{+\infty} E(0,t')\,\mathrm{d}t'. \tag{8}$$

Except for a constant factor, there is conservation of the pulse area whatever the pulse profile is. The second law only concerns the (causal) passive media. The conservation of the energy then results in the inequality

$$\int_{-\infty}^{t_0} |E(L,t')|^2\ \mathrm{d}t' \le \int_{-\infty}^{t_0} |E(0,t')|^2\ \mathrm{d}t' \tag{9}$$

valid for any t_0. Note however that, even in a purely dissipative medium, nothing prevents the instantaneous amplitude $|E(L,t')|$ of the output field to be larger and even much larger [15] than $|E(0,t')|$ at the same (retarded) time.

2.1 Group-velocity approximation and pulse-advancement

Following the method used to define the group velocity [1], we expand the complex phase $\Phi(\Omega)$ to the first order in Ω and we get

$$H(\Omega) \approx H(0)\exp\left[\mathrm{i}\Omega\ \Delta t(0)\right] \approx H(0)\exp\left\{\mathrm{i}\Omega\left[A(0) + \mathrm{i}B(0)\right]\right\} \tag{10}$$

where $\Delta t = \mathrm{d}\Phi/\mathrm{d}\Omega$, $A = \mathrm{d}\varphi/\mathrm{d}\Omega$ and $B = -\mathrm{d}F/\mathrm{d}\Omega$. The multiplication by $\exp(\mathrm{i}\Omega\,\Delta t)$ of the Fourier spectrum corresponds to a time-shift Δt for the signal. The envelope of the output pulse thus reads

$$E(L,t') \approx H(0)E\left[0, t' + A(0) + \mathrm{i}B(0)\right]. \qquad (11)$$

In order to facilitate further discussions, we now choose a reference frequency ω_0 attached to the dispersion region of the medium and we denote Ω_0 the deviation of the working frequency from ω_0. The complex time-shift then reads $\Delta t = A(\Omega_0) + \mathrm{i}B(\Omega_0)$. In our retarded time picture, its real part $A(\Omega_0)$ is the advancement of the output pulse on a pulse having covered the same distance at the luminal velocity. Since $\varphi(\pm\infty) = 0$, $\int_{-\infty}^{+\infty} A(\Omega_0)\mathrm{d}\Omega_0 = 0$. This result, common to every minimum phase system, ensures that there exist spectral regions where A is actually positive. One can easily verify that $A = L/c - L/v_g$, where v_g is the group velocity obtained by the standard procedure. In the experiments considered here [4,5,7–10], the group-delay is strongly negative and A is nearly equal to the absolute advancement $-L/v_g$. Due to the causal character of the complex gain-factor $\Gamma(\Omega)$, the gain factor $F(\Omega)$ and the phase $\varphi(\Omega)$ are a pair of Hilbert transforms and it is the same for their derivatives with respect to the frequency [16]. We then get

$$A(\Omega_0) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{d}F/\mathrm{d}\Omega}{\Omega_0 - \Omega}\mathrm{d}\Omega \qquad (12)$$

and integration per parts gives

$$A(\Omega_0) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{F(\Omega) - F(\Omega_0)}{(\Omega - \Omega_0)^2}\mathrm{d}\Omega. \qquad (13)$$

Equation (13) shows that only the spectral regions where $F(\Omega) > F(\Omega_0)$ contribute to the pulse-advancement whereas the regions where $F(\Omega) < F(\Omega_0)$ reduce this advancement. The largest advancements are expected when the gain of the medium presents a narrow minimum (preferably an absolute minimum) at the working frequency. Equations looking similar to equations (12, 13) have been obtained by Bolda *et al.* [17]. Note however that the integral given in their equation (17) diverges (double pole on the real axis), except when $\kappa(\omega)$ is strictly equal to zero (small is not sufficient), and that the term $(\mathrm{d}\kappa/\mathrm{d}\omega)/(\omega' - \omega)$ in their equation (20) is superfluous since it gives a zero contribution to the integral.

The imaginary part $B(\Omega_0)$ of the time shift Δt is detrimental to the observation of negative group velocity propagation. Associated to the 1st order variations of the medium gain with the frequency, its main effect is to shift the mean frequency of the pulse towards spectral regions of larger gain (or of lower absorption) and then to change the pulse-velocity during the propagation. By this means, it has even been possible to observe a transition from superluminal to subluminal velocity [6]. Except in the remarkable case of Gaussian pulses, an imaginary time-shift also results in a dramatic reshaping of the pulse-envelopes. Giving only two examples, a shift equal to $\mathrm{i}\pi/2$ transforms $1/\cosh(t)$ (sech-pulse) into $1/\sinh(t)$ and a double-humped profile as $t^2\exp(-t^2)$ is changed into a flat-topped one by a shift equal to i. In order to avoid any frequency-shift and pulse-reshaping (at least at this order of approximation), it is thus necessary that $B(\Omega_0)$ cancels. This condition is automatically fulfilled when the working frequency coincides with a minimum of the medium gain ($\mathrm{d}F/\mathrm{d}\Omega = 0$ for $\Omega = \Omega_0$), that is precisely when large (real) pulse-advancements A are expected. Taking the frequency of minimum gain as reference frequency (*i.e.* again $\Omega_0 = 0$), we then get

$$E(L,t') \approx H(0)E(0, t' + A) \qquad (14)$$

with

$$A = P \int_{-\infty}^{+\infty} \frac{F(\Omega) - F(0)}{\pi\Omega^2}\mathrm{d}\Omega. \qquad (15)$$

In these conditions, the advancement A has a very simple interpretation valid beyond the 1st order or group-velocity approximation. It may indeed be identified to the advancement of the centre of gravity of the pulse-envelope. Quite generally, the centre of gravity of a signal $f(t)$ of non-zero area may be defined by

$$\langle t \rangle = \frac{\int_{-\infty}^{+\infty} t\, f(t)\,\mathrm{d}t}{\int_{-\infty}^{+\infty} f(t)\,\mathrm{d}t} = \frac{\mathrm{i}}{\widehat{f}(0)}\left(\frac{\mathrm{d}\widehat{f}}{\mathrm{d}\omega}\right)_{\omega=0} \qquad (16)$$

where the last expression is derived from the so-called moment theorem (see for example pp. 16-17 in [13]). Applying this result to $E(L,t')$ and taking equations (3, 5) into account, we get

$$\langle t'(L) \rangle = \mathrm{i}\left(\frac{\mathrm{d}F}{\mathrm{d}\Omega}\right)_{\Omega=0} - \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\Omega}\right)_{\Omega=0} + \frac{\mathrm{i}}{\widehat{E}(0,0)}\left(\frac{\mathrm{d}\widehat{E}(0,\Omega)}{\mathrm{d}\Omega}\right)_{\Omega=0} = -A + \langle t'(0) \rangle. \qquad (17)$$

A actually appears as the advancement of the centre of gravity of the envelope of the output pulse over that of the incident pulse. This result is valid whatever the distortion is. It is especially useful when the envelope of the incident pulse (and thus of the output pulse) is purely real and provides in particular a good check of the numerical simulations.

Coming back to the group-velocity approximation (see Eq. (14)), we remark that it is quite possible to build a dispersive medium such that $F(0) = 0$ and $H(0) = 1$ (see the example given in Sect. 5). However, in all the experiments achieved up to now, the amplitudes of the input and output pulses were different and the difference was compensated by an external broadband (*i.e.* non-dispersive) amplification or attenuation. The corresponding device should be included in the system under consideration. The gain factor, transfer function and impulse response of the complete system (henceforth the system) are respectively $F'(\Omega) = F(\Omega) - F(0)$, $H'(\Omega) = H(\Omega)/H(0)$

and $h'(t') = h(t')/H(0)$. In the group velocity approximation, $H'(\Omega) \approx \exp(\mathrm{i}\Omega A)$ and the system delivers an output signal

$$E'(L,t') \approx E(0,t'+A). \tag{18}$$

2.2 Pulse-distortion

The propagation at the group velocity is always accompanied by some distortion, even when the pulse is ideally smooth. The constraints to satisfy for a low-distortion propagation are obviously more severe when the group velocity is negative. Such group velocities are attained when the width of the anomalous dispersion region is small compared to the mean pulse frequency and the first order calculation leading to the concept of group velocity is valid only if the spectral width of the pulse is itself narrow compared to the former one. It is evidently essential to check the consistency of this calculation by examining the effect of the higher order terms.

Although it is not indispensable, we now assume that the distribution of the lines given by equation (6) is symmetric with respect to the frequency ω_0 of the minimum of gain, to which is tuned the working frequency. This assumption results in some simplification of the calculations without significant loss of generality. The transfer function of the system is then such that $H'(-\Omega) = H'^{*}(\Omega)$ and its impulse response $h'(t')$ is purely real. The advantage of such an arrangement is that, for reasons of symmetry, the pulse-advancement is extremum at the frequency where the 1st order distortion cancels.

According to the previous assumption, $H'(\Omega)$ can be expanded in a power series of $\mathrm{i}\Omega$ with real coefficients or, equivalently, in a power series of $\mathrm{i}\Omega A$ with both real and dimensionless coefficients. At this step it is natural to refer the advancement A to a time τ_p characterising the duration of the pulse. In the case of bell-shaped pulses considered hereafter, we define τ_p as the half width of the amplitude profile of the pulse at 1/e of its maximum. For a Gaussian pulse, $\tau_p = (2\ln 2)^{-1/2}\tau_p^I = 0.85\tau_p^I$, where τ_p^I is the full width at half-maximum (FWHM) of the intensity profile. Introducing the relative advancement $a = A/\tau_p$, the dimensionless frequency $u = \Omega\tau_p$ and taking into account the first order result, the transfer function $H'(u)$ can be expanded under the form

$$H'(u) = \mathrm{e}^{\mathrm{i}ua}\left[1 + \sum_{n=2}^{\infty} D_n\,(\mathrm{i}ua)^n\right] \tag{19}$$

where $D_2 < 0$ (minimum of gain). In the time domain, the multiplication by $\mathrm{i}u$ results in a time-derivative with respect to the dimensionless time $\theta = t'/\tau_p$ and the signal delivered by the system is then

$$E'(L,\theta) \approx E(0,\theta+a) + \sum_{n=2}^{\infty} D_n a^n \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\left[E(0,\theta+a)\right]. \tag{20}$$

The distortion of the pulse is obviously given by the series in equations (19, 20).

Quite generally the distortion can be characterised by its uniform norm D_∞ or its root-mean-square D_{rms}

$$D_\infty = \frac{\|E'(L,\theta) - E(0,\theta+a)\|_\infty}{\|E(0,\theta)\|_\infty} \tag{21}$$

$$D_{\mathrm{rms}} = \left[\frac{\int_{-\infty}^{+\infty} |E'(L,\theta) - E(0,\theta+a)|^2\,\mathrm{d}\theta}{\int_{-\infty}^{+\infty} |E(0,\theta)|^2\,\mathrm{d}\theta}\right]^{\frac{1}{2}}. \tag{22}$$

Owing to the Parseval theorem, D_{rms} can also be directly expressed in terms of the transfer function

$$D_{\mathrm{rms}} = \left[\frac{\int_{-\infty}^{+\infty} \left|\left[H'(u) - \mathrm{e}^{\mathrm{i}ua}\right]\widehat{E}(0,u)\right|^2 \mathrm{d}u}{\int_{-\infty}^{+\infty} \left|\widehat{E}(0,u)\right|^2 \mathrm{d}u}\right]^{\frac{1}{2}}. \tag{23}$$

D_{rms} is only slightly smaller than D_∞ (factor close to unity) when the distortion is regularly distributed in the pulse profile. Significant deviations from this rule indicate either a localised defect or an extended distortion.

Except in particular cases [18], the convergence of the series in equations (19, 20) has not been analytically proved but numerical simulations show that a good approximation of the exact result is often given by restricting the series to its lowest order term ($n = 2$), if it is small enough. Equations (19, 20) then read

$$H'(u) \approx \mathrm{e}^{\mathrm{i}ua}\left(1 + \frac{\varepsilon}{2}u^2\right) \tag{24}$$

$$E'(L,\theta) \approx E(0,\theta+a) - \frac{\varepsilon}{2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left[E(0,\theta+a)\right] \tag{25}$$

with $\varepsilon = -2D_2a^2 > 0$. If $|D_2|$ can be made small enough, one may expect to observe significant advancement ($a \sim 1$) with low distortion ($\varepsilon \ll 1$).

Although the propagation at a negative group velocity is not specific to Gaussian pulses, these pulses are convenient for simple calculations. Putting $E(0,\theta) = \exp\left(-\theta^2\right)$ in equation (25) and doing some transformations valid at the 1st order in ε, we get

$$E'(L,\theta) \approx (1+\varepsilon)\exp\left[-(\theta+a)^2(1+\varepsilon)^2\right]. \tag{26}$$

Equations (21, 22) then gives $D_\infty = \varepsilon$ and $D_{\mathrm{rms}} = \varepsilon\sqrt{3}/2$. For this particular pulse-shape and at this order of approximation, the alteration of the pulse simply consists of a narrowing of the amplitude profile by the factor $\beta = (1+\varepsilon)$ and its magnification by the same factor, in agreement with the conservation of the envelope area (Eq. (8)). The intensity profile $|E'(L,\theta)|^2$ is also narrowed by the factor β but magnified by the factor β^2. The pulse-narrowing results in different advancements on the fall and the rise of the pulse, respectively $A_\downarrow = A + \tau_a - \tau_a/\beta$ and $A_\uparrow = A - \tau_a + \tau_a/\beta$, where τ_a indicates the half-width of the pulse at the arbitrary amplitude or intensity at which $A_\downarrow$ and $A_\uparrow$ are measured. The difference $(A_\downarrow - A_\uparrow)$

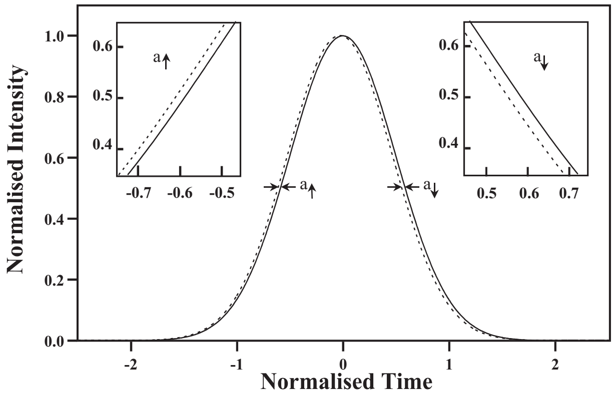


Fig. 1. Superluminal propagation of a Gaussian pulse in a case of modest advancement and low distortion. The intensity-profile of the output pulse (dotted line) is derived from equation (24) with $a = 0.03$ and $\varepsilon = 0.01$. The time unit is τ_p, the half width at 1/e of the amplitude-profile of the incident pulse. The intensity-profile of the incident pulse, delayed of the luminal propagation time, is given for reference (full line). The inserts show the difference of the advancements on the rise and on the fall of the pulse ($a_\uparrow = A_\uparrow/\tau_p$ and $a_\downarrow = A_\downarrow/\tau_p$). The values given to a and ε fit the experiment of Dogariu *et al.* [8].

is suitably characterised by the ratio

$$\xi = \frac{A_\downarrow - A_\uparrow}{A_\downarrow + A_\uparrow} \approx \frac{A_\downarrow - A_\uparrow}{2A} = \frac{\varepsilon\tau_a}{a\tau_p}. \tag{27}$$

ξ may take appreciable values when the relative advancement a is modest, even if the distortion is low. The experiment of Dogariu *et al.* with near Gaussian pulses provides an example of such a situation (see Fig. 6 in [8]). $A_\downarrow$ and $A_\uparrow$ are roughly in the proportion of 3 to 2 ($\xi \approx 0.2$) with $A = 63$ ns, $\tau_a = 1.20\ \mu$s, $\tau_p^I = 2.40\ \mu$s and $\tau_p = 2.04\ \mu$s, that is $a = 0.03$. By means of equation (27) it is possible to estimate the distortion parameter ($\varepsilon \approx 0.01$) without knowing the details of the experiment (examined in Sect. 4). As shown Figure 1, the difference between the advancements $A_\downarrow$ and $A_\uparrow$ observed by these authors is well retrieved on the analytic intensity-profile derived from our equation (26).

The previous calculations of the pulse-distortion implicitly assume that the pulse envelope is analytic and, in particular, that it has neither beginning nor end. As a matter of fact, a realistic pulse is time-limited and thus non-analytic. The unavoidable singularities associated to the beginning and the end of the pulse will be responsible of the appearance of transients patterns or wiggles superimposed to the slowly varying part of the envelope. These patterns are identified with or clearly related to the forerunners of Sommerfeld and Brillouin [1] and their wavefront propagates at the luminal velocity (see Eq. (7)). The general study of the phenomenon is beyond the scope of the present paper. It is sufficient to note here that, in well-conditioned experiments, its contribution to the distortion can be maintained at a very low level by lengthening the pulse pedestal and by eliminating the lowest order discontinuities, which originate the largest wiggles [18].

3 Single absorption-line arrangement

The single absorption-line arrangement provides the simplest and probably the most efficient way to obtain strongly negative group velocities. Fixing the reference frequency ω_0 at the resonance (absolute minimum of gain), the complex gain factor takes the simple form

$$\begin{aligned}\Gamma(\Omega) = F(\Omega) + \mathrm{i}\varphi(\Omega) &= -\frac{Z}{1+\mathrm{i}\Omega/\gamma} \\ &= -\frac{Z}{1+\Omega^2/\gamma^2} + \frac{\mathrm{i}Z\Omega/\gamma}{1+\Omega^2/\gamma^2}\end{aligned} \tag{28}$$

where $Z = \alpha L/2$ is the optical thickness on resonance for the field, α being the corresponding absorption coefficient for the intensity. The advancement A and the imaginary advancement B (responsible of a 1st order distortion) are immediately derived from equation (28)

$$A(\Omega_0) = \frac{Z(1-\Omega_0^2/\gamma^2)}{\gamma(1+\Omega_0^2/\gamma^2)^2} \quad B(\Omega_0) = -\frac{2Z\Omega_0}{\gamma^2(1+\Omega_0^2/\gamma^2)^2}. \tag{29}$$

As expected, $B = 0$ on resonance and the real advancement A is then maximum (see Fig. 2). According to our discussion of equation (13), this simply results from the fact that all the frequencies then contribute to the advancement. Let us remark that the advancement A on resonance is much larger than the delays (negative advancements) occurring in the wings of the line and cancels out when $|\Omega_0| = \gamma$. In these latter points, corresponding to the half-width at half-maximum of the line profile, there is a transition from a superluminal to a subluminal group velocity [6,19].

When the choice of the working frequency is optimal ($\Omega_0 = 0$), $A = Z/\gamma$ and $H(0) = \mathrm{e}^{-Z}$. In the experiments, an external broadband (non-dispersive) amplifier of amplitude-gain e^Z compensates the medium absorption. Introducing again the dimensionless quantities $a = A/\tau_p$ and $u = \Omega\tau_p$, the transfer function of the complete system may be written

$$H'(u) = \exp\left(\frac{\mathrm{i}ua}{1+\mathrm{i}ua/Z}\right) = e^{\mathrm{i}ua}\exp\left(\frac{-(\mathrm{i}ua)^2/Z}{1+\mathrm{i}ua/Z}\right). \tag{30}$$

In the low distortion limit, the transfer function and the output signal take the general form given by equations (24, 25) with $\varepsilon = 2a^2/Z$. Coming back to the original variables, $\varepsilon = 2Z/\gamma^2\tau_p^2$ and the condition of low distortion reads $\tau_p \gg \sqrt{2Z}/\gamma$. Significant pulse-advancements are only obtained for large optical thickness Z and the previous condition is more severe than that usually considered ($\tau_p \gg 1/\gamma$). Conversely, for a given relative advancement a, the distortion can be made as low as wanted by taking a large enough optical thickness. For comparison with other arrangements, it is convenient to relate the relative advancement a, the distortion parameter ε (assumed to be small) and the peak value P of the amplitude gain $|H'(u)|$ of the system. In the present case,

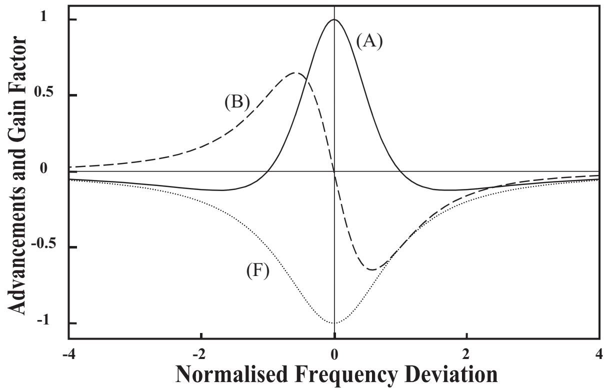


Fig. 2. Single absorption-line arrangement. Dependence of the effective advancement A (full line), of the imaginary advancement B (dashed line) and of the medium gain-factor F (dotted line) on the frequency deviation from resonance. The advancements, gain factor and frequency deviation are normalised to Z/γ, Z and γ respectively.

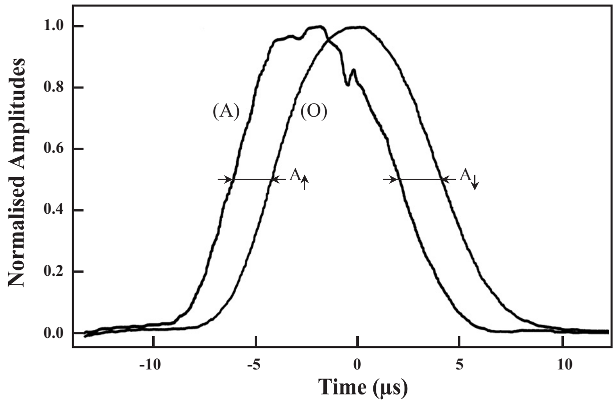


Fig. 3. Experimental evidence of propagation at a negative group velocity in a resonant molecular absorber [5]. The pulse (A) having propagated in the gas seems to arrive about 2 μs earlier than the pulse (O) having covered the same distance ($L = 24$ m) in the empty cell. The detection-chain gains used to obtain the pulses (A) and (O) are in a ratio of about 190.

this peak value, attained for $\Omega \to \pm\infty$, is obviously e^Z and we get

$$P = \exp\left(\frac{2a^2}{\varepsilon}\right). \tag{31}$$

Similar expressions will be obtained with the gain-doublet arrangement (Sect. 4) when the 2nd order distortion prevails. A preliminary conclusion is that significant advancements ($a \sim 1$) with low distortion ($\varepsilon \ll 1$) can be attained only in systems with large peak gain.

The first experimental evidence of significant pulse-advancements associated to a negative group velocity has been achieved by Chu and Wong [4]. They used 9.5 or 76 μm thick samples of epitaxially grown GaP:N, a pulsed dye laser tuned in the vicinity of the well isolated bound A-exciton line at 534 nm and detection involving a 2nd order correlation technique. The parameters of a typical experiment (see Figs. 2 and 3a in [4] and [20,21]) are $Z \approx 2.85$ ($P \approx 17$), $\gamma \approx 1.23 \times 10^{11}$ Rad s^{-1} ($1/\gamma = 7.8$ ps), $\tau_p^I = 34$ ps and $\tau_p \approx 29$ ps (value estimated by assimilating the pulses to Gaussian ones). A pulse-advancement $A \approx 22$ ps, in good agreement with the relation $A = Z/\gamma$, was observed but this considerable advancement ($a \approx 0.76$) is accompanied by a strong narrowing of the pulse [20,21], roughly by a factor 1.8. The condition of low distortion is indeed poorly fulfilled ($\varepsilon \approx 0.41$). Numerical simulations with the previous parameters show that, besides the above-mentioned narrowing, the pulse is affected by an appreciable asymmetry, which could not be revealed by the detection technique used in the experiment [20].

The pioneering work of Chu and Wong urged us to undertake experiments involving *a real-time detection of the true shape of the field envelope* and complying with the conditions of low distortion. Our experiments [5] were achieved on a low-pressure gas at a wavelength $\lambda \approx 3$ mm where several molecules present well-isolated strong rotation lines. The gas pressure ranged from 7 to 30 Pa, a domain where the linewidth γ is mainly determined by the molecular collisions and is proportional to the pressure. The gas was enclosed in a 24 m-long cell and it was proceeded to a *direct* comparison of the field envelope of the output pulses obtained with and without gas. The envelope of the incident field was generated from a signal of the form $(1 + \cos\sigma t)$ for $-\pi/\sigma < t < +\pi/\sigma$ (0 elsewhere), suitably reshaped and lengthened in its pedestal in order to eliminate the strong wiggles which would result from the discontinuities of the 2nd derivative at $t = \pm\pi/\sigma$. Figure 3, obtained from the original data of [5], shows the results of a typical experiment. The parameters are $Z \approx 5.25$ ($P \approx 190$), $1/\gamma \approx 0.35$ μs and $\tau_p \approx 4.8$ μs. The observed advancement $A \approx 2.0$ μs is much larger than the luminal propagation time ($L/c = 80$ ns) and in good agreement with its theoretical value ($Z/\gamma = 1.84$ μs). The relative advancement a is significant ($a = 0.42$) and the distortion parameter ε, inferred from Z and a, is small ($\varepsilon = 0.067$). The distortion is actually low. The difference between the advancements $A_\downarrow$ and $A_\uparrow$ at half-maximum on the fall and the rise of the pulse is even smaller than it would be in the case of a Gaussian pulse of same duration τ_p, with $\xi \approx 5\%$ instead of 14% (value derived from Eq. (27)). This simply results from the particular shape of the incident pulse. Its main part is indeed nearly sinusoidal and is then negligibly distorted owing to the system linearity, the distortion being more important in the non-sinusoidal wings. The envelope (A) is however slightly distorted in its central part by wiggles, which originate from very small discontinuities in the amplitude-profile of the incident pulse. As it happens with piecewise function-generators, such a discontinuity occurs in particular near the maximum of the modulation signal and is responsible of the most visible wiggle. Its location below the maximum of the pulse (O) having propagated in vacuum confirms that the propagation of localised defects is actually luminal.

In agreement with the theoretical analysis (see Eq. (31)), the significant pulse-advancement with low distortion obtained in the previous experiment is paid at the price of a large absorption of the medium at the mean frequency of the pulse or, equivalently, of a large peak gain P of the complete system. Conversely, more modest advancements can be attained in nearly transparent media. For example the result shown Figure 1 ($a = 0.03$, $\varepsilon = 0.01$) would be obtained in a medium of amplitude-gain $\exp(2a^2/\varepsilon) \approx 0.835$, corresponding to an intensity transmittance as high as 70%.

To close this section, let us mention that very large anomalous dispersions have been recently demonstrated at optical wavelength in a coherently prepared atomic vapour [10]. The dispersion originates from an electromagnetically induced absorption of subnatural width. This arrangement provides the opportunity of observing negative group velocity propagation with long laser-pulses.

4 Gain-doublet arrangement

The proposal of exploiting the anomalous dispersion occurring between two gain-lines in order to observe negative group velocity propagation has been made for the first time by Steinberg and Chiao [22]. Note however that the model developed by these authors is not very realistic. The refractive index of the atomic vapour is indeed assumed to be purely real (no gain) whereas the gain frequency-dependence is in fact the main cause of pulse-distortion in the experiments.

Fixing the reference frequency ω_0 halfway between the two gain-lines, the complex gain factor of the medium reads

$$\Gamma(\Omega) = F(\Omega) + \mathrm{i}\varphi(\Omega) = \frac{G}{1 + \mathrm{i}\Delta + \mathrm{i}\Omega/\gamma} + \frac{G}{1 - \mathrm{i}\Delta + \mathrm{i}\Omega/\gamma} \tag{32}$$

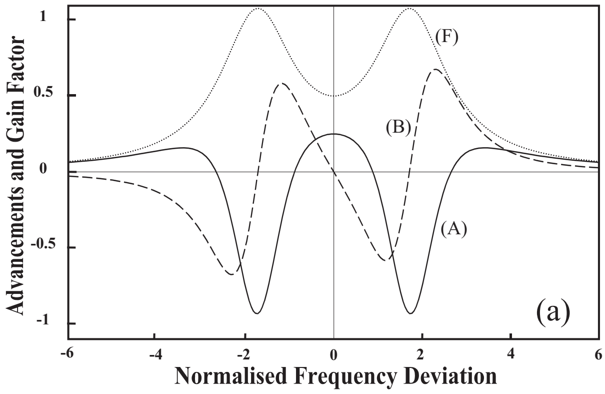
where $G = gL/2 > 0$ is the amplitude-gain factor of each line on resonance and $\Delta = (\omega_2 - \omega_1)/2\gamma > 0$. The gain-factor F is obviously an even function of Ω and it is easily shown that a gain-doublet appears as soon as the splitting parameter Δ exceeds $1/\sqrt{3}$. The two maximums occur at $\Omega = \pm\Omega_{\max}$, with

$$\Omega_{\max} = \gamma\left(2\Delta\sqrt{1+\Delta^2} - 1 - \Delta^2\right)^{\frac{1}{2}}. \tag{33}$$

The amplitudes of the central minimum and of the maximums are respectively

$$F_{\min} = F(0) = \frac{2G}{1+\Delta^2}, \quad F_{\max} = \frac{G}{2\Delta}\left(\Delta + \sqrt{1+\Delta^2}\right). \tag{34}$$

Figure 4 shows the frequency-dependence of the gain factor F, of the effective advancement A and of the imaginary advancement B for two representative values of Δ, respectively $\sqrt{3}$ and 4. As expected, B cancels and A is extremum at the central frequency. The extremum of A is a maximum (as in the single absorption-line case) for $\Delta = \sqrt{3}$ but is a relative minimum for $\Delta = 4$. Let us remark that in both cases the advancement is much smaller than the delays occurring in the vicinity of the two maximums of gain. When the working frequency is tuned to the reference frequency, the pulse advancement takes the simple form



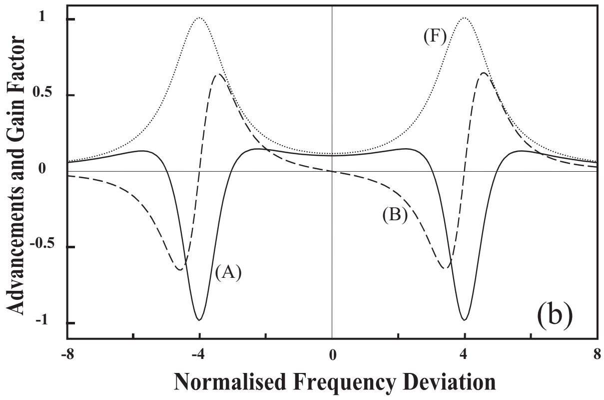


Fig. 4. Gain doublet arrangement with (a) $\Delta = \sqrt{3}$ and (b) $\Delta = 4$. Dependence of the effective advancement A (full line), of the imaginary advancement B (dashed line) and of the medium gain-factor F (dotted line) on the deviation from the reference frequency. The advancements, gain factor and frequency deviation are normalised to G/γ, G and γ respectively.

$$A = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}\Omega}\right)_{\Omega=0} = \frac{2G}{\gamma}\frac{\Delta^2 - 1}{(\Delta^2+1)^2}. \tag{35}$$

There is actually an advancement when $\Delta > 1$ and, if G and γ are fixed, the advancement is maximum for $\Delta = \sqrt{3}$ (Fig. 4a). In the experiments, the medium gain $H(0) = \exp(F_{\min})$ is compensated by a non-dispersive attenuation and the complete system has a peak gain P

$$P = \exp(F_{\max} - F_{\min}). \tag{36}$$

By combining equation (32) and equation (35), its transfer function is easily expressed as a function of the dimensionless frequency u and of the parameters a, Δ and G

$$H'(u) = \exp[\Gamma(u) - F_{\min})] = \exp\left[\mathrm{i}ua\frac{N(u)}{Q(u)}\right] \tag{37}$$

with

$$N(u) = 1 - \frac{\mathrm{i}ua}{2G}\frac{\left(\Delta^2+1\right)^2}{\left(\Delta^2-1\right)^2} \tag{38}$$

$$Q(u) = 1 + \frac{\mathrm{i}ua}{G}\frac{\left(\Delta^2+1\right)}{\left(\Delta^2-1\right)} + \frac{(\mathrm{i}ua)^2}{4G^2}\frac{\left(\Delta^2+1\right)^3}{\left(\Delta^2-1\right)^2}\cdot \tag{39}$$

By expanding $H'(u)$ to the 2nd order in u, one gets the 2nd order distortion parameter ε

$$\varepsilon = \frac{\left(\Delta^2+1\right)\left(3\Delta^2-1\right)}{G\left(\Delta^2-1\right)^2}a^2 \tag{40}$$

and the system peak-gain may be rewritten under a form similar to that obtained in the single-line case

$$P = \exp\left(\frac{2a^2}{\varepsilon}r(\Delta)\right) \tag{41}$$

with

$$r(\Delta) = \frac{\left(\Delta^2+1\right)\left(3\Delta^2-1\right)}{\left(\Delta^2-1\right)^2}\left(\frac{\Delta+\sqrt{\Delta^2+1}}{4\Delta} - \frac{1}{\Delta^2+1}\right). \tag{42}$$

Equation (40) shows that the 2nd order distortion never cancels when there is a pulse advancement ($\Delta > 1$). The gain-line e^G and the system peak-gain P required to attain a given relative advancement a for a fixed value of ε are decreasing functions of Δ. Both tend to $\exp(3a^2/\varepsilon)$ for $\Delta \gg 1$, that is when the two components of the doublet are well separated.

The equations linking the peak gain P, the relative advancement a and the 2nd order distortion parameter ε in the single absorption-line scheme (Eq. (31)) and in the gain-doublet arrangement (Eqs. (41, 42)) are exact but they are only meaningful if the 2nd order distortion constitutes a good approximation of the exact distortion. In fact the condition $\varepsilon \ll 1$ is not always sufficient. To clarify this point, we again consider the case of Gaussian pulses. If the 2nd order approximation is valid, the uniform-norm and rms distortions are $D_\infty \approx D_\infty^{(2)} = \varepsilon$ and $D_{\mathrm{rms}} \approx D_{\mathrm{rms}}^{(2)} = \varepsilon\sqrt{3}/2$ respectively (see Sect. 2.2). There is no problem with the single absorption-line arrangement. It has indeed be proven [18] that, if $\varepsilon < 1/2$,

$$D_\infty < \frac{\varepsilon}{1-2\varepsilon} \tag{43}$$

D_∞ is actually very close to ε when $\varepsilon \ll 1$. With the parameters of the experiment described in Section 3 ($a = 0.42$, $\varepsilon = 0.067$), equation (43) leads to $D_\infty < 1.15\varepsilon$ and a numerical simulation shows that $D_\infty = 1.09\varepsilon = 0.073$. This means that the 2nd order approximation reproduces the exact solution with a precision better than 0.6%. With the gain-doublet arrangement on the contrary, large discrepancies appear when the separation of the two lines is large ($\Delta \gg 1$). Due to the lack of an inequality comparable to equation (43), it is useful to examine the higher order terms in the expansion of the transfer function (see Eq. (19)). From equations (37–39), one easily obtains the coefficient of the 3rd power term

$$D_3 = -\frac{\left(\Delta^2+1\right)^2\left(\Delta^4-6\Delta^2+1\right)}{4G^2\left(\Delta^2-1\right)^3}\cdot \tag{44}$$

We incidentally remark that the 3rd order distortion cancels out ($D_3 = 0$) for $\Delta = 1+\sqrt{2}$ and that the corresponding curve $A(\Omega)$, intermediate between those shown Figure 4, is flat-topped in $\Omega = 0$ where $\mathrm{d}^2A/\mathrm{d}\Omega^2 = \mathrm{d}^3\varphi/\mathrm{d}\Omega^3 \sim D_3$. Numerical simulations show that the 2nd order approximation is satisfactory for $\varepsilon \ll 1$ when Δ is smaller than or comparable to this value $1+\sqrt{2}$ (say up to $\Delta = 4$). If on the contrary $\Delta \gg 1$, equation (44) shows that D_3 is proportional to Δ^2 and, more generally, one can deduce from equations (37–39) that D_{2n+1} and D_{2n+2} are proportional to Δ^{2n} for $n > 0$. Even for small ε, the expansion of $E'(L,\theta)$ given by equation (20) then fails to converge. A numerical calculation achieved for $\Delta = 7$, $a = 0.42$ and $\varepsilon = 0.067$, shows that $D_\infty = 80\%$ and $D_{\mathrm{rms}} = 155\%$, instead of 6.7% and 5.8% in the 2nd order approximation. The large ratio $D_{\mathrm{rms}}/D_\infty$ (close to 2) indicates an extended distortion. The *extra-distortion* in fact consists of an oscillation at a frequency equal to the half-separation of the two components of the gain-doublet (see Fig. 5). The relative amplitude of the Fourier spectrum of the incident pulse is small at the frequencies of the two lines ($\approx 8\times10^{-4}$) but, on account of the corresponding gain ($P > 3\,000$), is indeed sufficient to excite transient oscillations at the eigenfrequencies of these narrow-band amplifiers. This interpretation is confirmed by the values of the rise and fall times of the oscillations, roughly equal to the inverse of the half-width at half-maximum of each peak in the system gain (see insert in Fig. 5). The phenomenon is especially apparent for large Δ because the

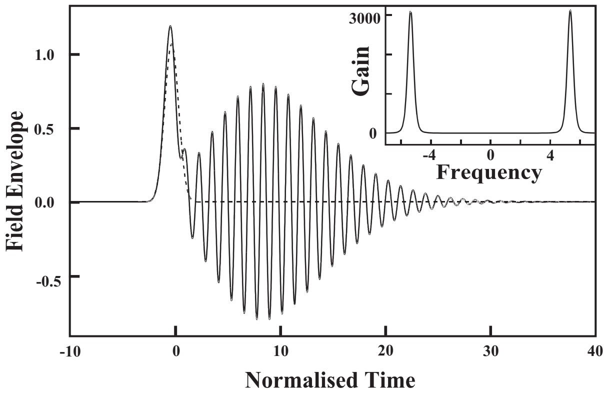


Fig. 5. Gain doublet arrangement. Numerical evidence of the extra-distortion occurring for a large doublet splitting. The envelope of the output field (full line) departs from the result obtained in the 2nd order approximation (dotted line) by a strong oscillation. The time unit is τ_p, the half width at 1/e of the envelope of the Gaussian incident pulse. The parameters are $\Delta = 7$, $G = 8.34$ and $\gamma\tau_p = 0.763$, resulting in $a = 0.42$ and $\varepsilon = 0.067$. The insert displays the spectral profile of the system amplitude-gain $|H'(u)|$. The frequency unit is $1/\gamma_p$.

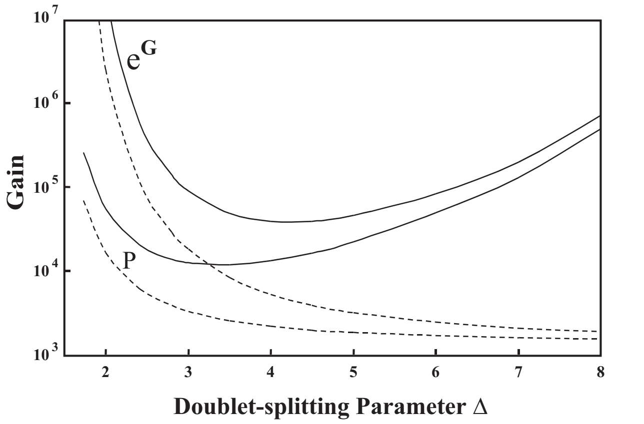


Fig. 6. Gain-doublet arrangement. Line amplitude-gain e^G and peak amplitude-gain P required to attain $a = 0.42$ with $D_\infty = 0.073$, as functions of the doublet-splitting parameter Δ (full lines). The dotted lines show the analytical result obtained when this is the 2nd order distortion (instead of the exact distortion) which is fixed to 0.073.

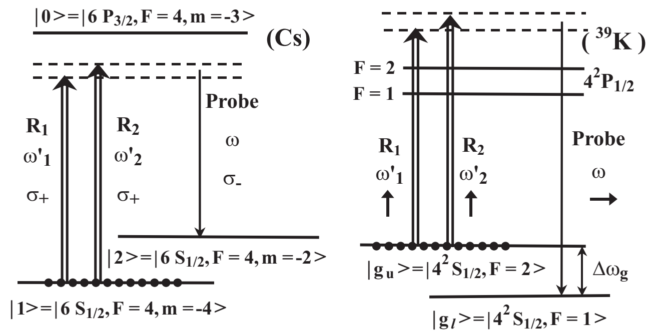


Fig. 7. Level schemes used to achieve a gain doublet in a vapour of caesium (left) and of potassium 39 (right). For sake of visibility, the figures are not drawn to scale and only the most relevant levels are shown. The dotted lines indicate the virtual levels induced by the Raman pumps R_1 and R_2. The symbols (σ_+, σ_-) and ($\uparrow$, $\rightarrow$) specify the polarisation of the different light-beams.

advancement in absolute value A decreases when Δ increases (see Eq. (35) and compare Figs. 4a and 4b) and it is necessary for keeping the same relative advancement a to reduce the duration τ_p of the pulse. Its spectrum then takes non-negligible amplitudes at the lines frequencies and the oscillations previously described occur. From this viewpoint, the Gaussian pulses are privileged by the rapid fall of the wings of their spectrum. More dramatic effects are expected with other analytic pulses (*e.g.* sech-pulses) and, *a fortiori*, with pulses whose envelope present singularities (even slight).

Due to the extra-distortion, the line-gain e^G and the system peak-gain P required to attain a given advancement a with a fixed distortion D_∞ do not take their minimum values for $\Delta \gg 1$, as it would occur if the 2nd order distortion would dominate (see Eqs. (40–42)), but for intermediate values of Δ. Figure 6 shows the result of a numerical research of these minimums when a and D_∞ have the same values as in the reference single-line experiment, respectively 0.42 and 0.073. The minimum occurs at $\Delta \approx 3.5$ for the system peak-gain and at $\Delta \approx 4.2$ for the line-gain. The corresponding values of ε (see Eq. (40)) are respectively 0.061 and 0.059, not far below the exact distortion D_∞ (0.073). The main part of the distortion then originates from the 2nd order term. However, the line-gain and the system peak-gain are very sensitive to the reference distortion and largely exceed the analytical values obtained by taking $\varepsilon = D_\infty$ (see dotted lines in Fig. 6). An important point is that the minimum of the peak-gain ($\approx 1.2 \times 10^4$) is considerably larger than the peak-gain required in the single absorption-line arrangement ($\approx$190) to attain the same advancement a with the same distortion D_∞. This result is in full agreement with our analysis of equation (13). In the single absorption-line case, $F(\Omega) > F(0)$ whatever Ω is (see dotted line in Fig. 2) and all the frequencies contribute to the advancement. Conversely, the spectral regions where $F(\Omega) < F(0)$ in the gain-doublet scheme (see Fig. 4) give a negative contribution to the advancement (Eq. (15)) which should be compensated by a correlative increase of the peak gain. All these results are confirmed by simulations made for other values of a and D_∞.

Dogariu, Kuzmich and Wang [7,8] have achieved the first experiment with a gain-doublet arrangement in the near infrared. Their medium is an atomic caesium (Cs) vapour ($L = 6$ cm) and the gain-doublet is obtained by a Raman technique. Figure 7 (left part) shows the relevant energy levels. The Cs atoms are protected against the Earth's magnetic field by a suitable shield and submitted to a uniform magnetic field ($\sim 10^{-4}$ T) parallel to the light propagation direction, serving the purpose of removing the Zeeman degeneracy. Two cw lasers are used to optically pump almost the Cs atoms in the ground-state hyperfine level $|1\rangle = \left|6S_{1/2},\ F = 4,\ m = -4\right\rangle$ and to empty the final state of the Raman transition $|2\rangle = \left|6S_{1/2},\ F = 4,\ m = -2\right\rangle$. The Raman pumping is achieved by two cw beams R_1 and R_2 of frequencies ω_1' and ω_2', both right-hand circularly polarised and detuned to the red of the transition $|1\rangle \longrightarrow |0\rangle$. The probe beam is left-hand polarised. If only one beam R_n ($n = 1$ or 2) is applied, a maximum of gain is obtained at the probe frequency ω_n such that $(\omega_n' - \omega_n)$ coincides with the frequency of the forbidden transition $|2\rangle \longrightarrow |1\rangle$ (condition of 2-photon resonance). The pump and probe fields propagating in the same direction and the frequency differences $(\omega_n' - \omega_n)$ being very small, there is a nearly exact compensation of the Doppler effect. When the two beams are present, one obtains the expected double-humped gain profile (see Fig. 4). To demonstrate negative group velocity propagation, Dogariu *et al.* make use of near Gaussian probe-pulses and compare the intensity profile of the output pulse to that obtained when the working frequency is shifted far from resonance. Note incidentally that an intensity-detection, owing to its non-linear character, may conceal defects present in the pedestal of the field envelope. The parameters of a typical experiment

[8] are $\tau_p^I = 2.4$ μs (that is $\tau_p = 2.04$ μs), $G = 0.7$, $\gamma/2\pi = 0.45$ MHz and $(\omega_2' - \omega_1')/2\pi = (\omega_2 - \omega_1)/2\pi = 2.8$ MHz ($\Delta = 3.1$). By changing i in −i in the complex expressions (different convention in the complex notations) and taking into account that the frequencies considered throughout the present paper are angular frequencies, the expressions given by Dogariu *et al.* for the complex susceptibility, the pulse-advancement and the 2nd order pulse-distortion are strictly equivalent to ours, their parameter M being linked to G by the relation $M = c\gamma G/\pi\omega_0 L$. Applying these results to the experiment under consideration, we find a pulse-advancement $A \approx 38$ ns, a 2nd order distortion parameter $\varepsilon \approx 0.002$, a gain-factor contrast $F_{\max}/F(0) \approx 5.5$ and a separation of the gain maximums $2\Omega_{\max}/2\pi \approx 2.8$ MHz (see Eq. (33)), whereas their experimental values (see Figs. 5 and 6 in [8] and our Eq. (27)) are respectively 63 ns, 0.01, 3.0 and 2.5 MHz. We attribute the differences to inhomogeneous effects affecting in particular the relaxation rate γ. In order to achieve an efficient Raman pumping, the frequencies of the beams R_n are indeed not far from that of the transition $|1\rangle \longrightarrow |0\rangle$ (detuning comparable to the Doppler linewidth). The effective detuning (detuning in the atom frame) and thus the relaxation rate γ then depends on the atom velocity. Equal to the transit time broadening γ_t ($\gamma_t/2\pi \approx 0.35$ MHz) for large detuning, the relaxation rate may approach the excited-state decay rate γ_e ($\gamma_e/2\pi \approx 5.3$ MHz) for the atoms closer to resonance. The resulting spreading of the splitting parameter Δ ($\Delta \sim 1/\gamma$), which takes in particular values smaller than 3.1, explains both the reduction of the contrast and of the separation of the maximums in the gain profile and the increase of the advancement and of the distortion. The overlapping of the Doppler profile with the frequencies of the beams R_1 and R_2 has an important other consequence. There indeed exist atoms whose velocity is such that they are on resonance or quasi-resonance with R_1 and/or R_2. The Raman pumping then tends to deplete the level $|1\rangle$ and thus oppose to the optical pumping. According to Dogariu *et al.* [8], *the atoms reversely pumped away in these velocity groups act like a broadband weak absorber that helps to compensate the residual gain.* In fact, the absorption overcompensates the gain and the intensity-transmittance of the probe pulse is only 40%. Let us recall that the same relative pulse-advancement for a same level of distortion would be obtained with a 70% transmittance in the single absorption-line arrangement (see Sect. 2).

In the experiments of Dogariu *et al.*, the pulse advancements A are much larger than the luminal transit time ($L/c = 0.2$ ns) but they keep quite modest with respect to the pulse width (see Fig. 4 in [7] and Fig. 6 in [8]). The relative advancement a is only a few percents. As shown in detail in our theoretical study, this is the consequence of the small values of the line gain-factor ($G \approx 0.7$) and of the system peak-gain ($P \approx 1.5$). Gauthier and Stenner recently achieved experiments involving larger gains [9]. Their arrangement is similar to the previous one. The medium is an atomic vapour of potassium 39 (^{39}K) whose relevant energy levels are shown Figure 7 (right). The states concerned by the Raman transition are the states $|g_u\rangle = |4^2\mathrm{S}_{1/2},\ F=2\rangle$ and $|g_l\rangle = |4^2\mathrm{S}_{1/2},\ F=1\rangle$. A specific feature of their experiment is that the Raman pumps R_1 and R_2 also achieve the optical pumping. For this purpose, they are both detuned to the blue of the D_1 line of ^{39}K. Because their frequencies are closer to resonance with the transitions starting from the lower ground state $|g_l\rangle$, the population is preferentially pumped out of this state and into the upper ground state $|g_u\rangle$. Thanks to this arrangement, intense Raman pumps can be used without risk of depopulating the state $|g_u\rangle$ and high probe gains become possible. Again the gain occurs when the condition of 2-photon resonance is fulfilled, that is at probe frequencies greater than the pumps frequencies by about the ground state splitting ($\Delta\omega_g/2\pi \approx 460$ MHz). The parameters of a typical experiment [9,23] are $gL \approx 9.5$ that is $G \approx 4.75$, $(\omega_2' - \omega_1') \approx 24$ MHz and $\Delta \approx 2.41$. We derive from these data $1/\gamma \approx 31$ ns, $\mathrm{e}^G \approx 116$, $H(0) \approx 5$ and $P \approx 28$. The incident probe-pulse is flat-topped with $\tau_p^I \approx 184$ ns and the intensity profiles of the output pulses with and without potassium in the cell are directly compared. Gauthier and Stenner observed that the output pulse is then composed of a number of short pulses (7 are clearly visible), with a distance between two successive pulses of about 42 ns, very close to $2\pi/(\omega_2' - \omega_1')$. The phenomenon is interpreted in terms of four-wave mixing. In this process, the combination of the pump frequencies ω_1' and ω_2' with the probe frequency ω generates *at the lowest order in the probe field* a comb made of the frequencies $\omega + n(\omega_2' - \omega_1')$, which leads to the observed temporal structure. The new frequencies ($n \neq 0$) and the original frequency ω exponentially grow but the former start out with zero intensity. Their intensity keeps small when the gain of the medium is modest as in the experiments of Dogariu *et al.* [7,8]. Their effect is then negligible and the linear theory applies. Conversely, the sidebands play a prominent role when the gain is large (as required to attain significant pulse-advancements) and originate the pulse splitting experimentally evidenced by Gauthier and Stenner. The phenomenon, interesting from the viewpoint of nonlinear optics, obviously condemns the bichromatic pumping in the context of the present study because it leads to a large pulse reshaping. A gain-doublet can fortunately be created without generation of sidebands by using a monochromatic Raman pumping and a level-scheme involving a doublet of final states. The response of the medium is then correctly described by our purely linear theory (Eq. (32)) whatever the gain is. Figure 8 shows the output-pulse profiles, which would be obtained in conditions identical to those of Gauthier and Stenner. The incident field envelope is suitably modelled by an hyperGaussian function $\exp\left(-t^4/\tau_p^4\right)$ with $\tau_p = (8\ln 2)^{-1/4}\tau_p^I \approx 120$ ns. The oscillations pointed out in the analysis of the higher order distortion (see Fig. 5) now appear in the output field envelope for a moderate value of Δ and occur both after and during the incident pulse. The uniform norm and rms distortions are respectively $D_\infty \approx 41\%$ and $D_{\mathrm{rms}} \approx 33\%$. The modulation of the output field appearing in our fully linear model should

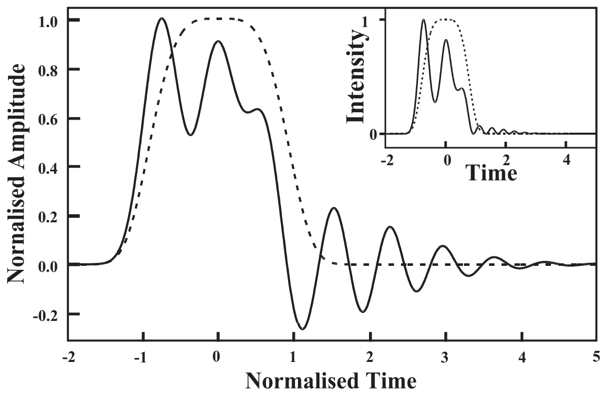


Fig. 8. Envelope of the output field (full line) numerically obtained with the parameters of the experiment of Gauthier and Stenner [9]. The envelope of the pulse having propagated in vacuum is given for reference (dotted line). The time unit is τ_p, the half width at 1/e of the amplitude-profile of the incident pulse and $a = 0.25$. The intensity-profiles shown in the insert illustrate the effects of a quadratic detection (exaggeration of the oscillations close to the maximum, considerable reduction of the amplitude of the post-oscillations and doubling of their frequency).

be carefully distinguished from that resulting from four waves mixing in the experiment of Gauthier and Stenner. It is nearly sinusoidal (not pulsed), its period is twice as long and, above all, it can be suppressed by using long enough incident pulses. The oscillations are in fact excited by the steep rise and fall of the incident-pulse envelope. They disappear when a Gaussian pulse of same duration τ_p is used in place of the flat-topped incident pulse and the distortions become then quite reasonable ($D_\infty \approx 7.6\%$ and $D_{\rm rms} \approx 6.3\%$), close to their 2nd order values ($D_\infty^{(2)} = \varepsilon \approx 6.4\%$ and $D_{\rm rms}^{(2)} \approx 5.5\%$). Incidentally, our simulations show the great sensitivity of the distortion to the shape of the incident pulse. When the distortion is as large as that shown Figure 8, the location of the pulse-maximum is not really meaningful but our analytic expression of the pulse-advancement (Eq. (35)) keeps exact for the centre of gravity of the pulse-envelope (see Eq. (17)).

5 Alternative arrangements

The main obstacle to the experimental evidence of significant pulse-advancements with moderate distortion lies in the catastrophic peak-values attained by the gain of the corresponding systems. Large gains at certain frequencies indeed result in a hypersensitivity of the experiments to unavoidable slight defects in the incident pulse-profile (see Fig. 3) and raise serious problems of noise and of instability. In optical systems, the transmitted probe-pulse may in particular be completely obscured by the amplified spontaneous emission [9] or undesired laser effects [22]. This entails that, among the systems achieving in principle the same relative advancement a with the same distortion D_∞, those having the smallest peak-gain P are the most suitable for real experiments. Alternatively, one may consider that, if P is fixed, the most efficient systems are those leading to the lowest distortion for a given advancement or to the largest advancement for a given distortion. We have pointed out in the previous section that, from this viewpoint, the single absorption-line arrangement is much more efficient than the gain-doublet arrangement. We show in the present section that its efficiency can be equaled in an amplifying medium whose gain falls to zero at the working frequency and which is thus purely transparent at this frequency ("gain-zero arrangement"). We then study the possibility of attaining larger efficiencies with an absorption-doublet arrangement when the 2nd order distortion cancels. We finally discuss the opportunity of developing more complex arrangements in order to cancel the distortion beyond the 2nd order.

5.1 Gain-zero arrangement

In the experiments analysed in Sections 3 and 4, the systems comprise an element exterior to the medium serving the purpose of compensating the absorption [4,5,7,8] or the gain [9] of the medium at the centre of the incident-pulse spectrum. In fact the function of this extra element can be ensured by the medium itself. It suffices in principle to use an amplifying medium with a dip in its gain-profile where the gain falls to zero. The medium is then purely transparent (neither amplifying nor absorbing) at this frequency which is obviously taken as working frequency. This could be achieved, *e.g.*, in a monochromatically driven Raman amplifier with a level scheme such that destructive quantum interferences cancel the gain at a particular frequency [24].

The previous situation is conveniently modelled by associating a gain-line and an absorption line having the same frequency and gain modulus but different widths. The complex gain-factor of the medium then reads

$$\Gamma(\Omega) = F(\Omega) + \mathrm{i}\varphi(\Omega) = -\frac{Z}{1+\mathrm{i}\Omega/\gamma} + \frac{Z}{1+\mathrm{i}\beta\Omega/\gamma} \quad (45)$$

where β is the ratio of the widths of the absorption and gain lines ($\beta < 1$). The profile of the gain-factor $F(\Omega)$ has a zero minimum at $\Omega = 0$ and two absolute maximums at $\Omega = \pm\gamma/\sqrt{\beta}$. The amplitude $F_{\max}$ of the latter and the peak gain P of the system (here reduced to the medium) are

$$F_{\max} = Z\frac{1-\beta}{1+\beta} \qquad P = \exp(F_{\max}). \quad (46)$$

The pulse advancement is simply the difference of the advancement Z/γ due to absorption-line and of the delay $\beta Z/\gamma$ due to the gain-line

$$A = \frac{Z}{\gamma}\left(1-\beta\right). \quad (47)$$

By combining equations (45–47), the transfer function of the system can be expressed as a function of the dimensionless quantities u, a, P and β. One gets

$$H'(u) = H(u)$$
$$= e^{iua} \exp\left(-\frac{(iua)^2/\ln P + \beta(iua)^3/(1+\beta)^2\ln^2 P}{1 + iua/\ln P + \beta(iua)^2/(1+\beta)^2\ln^2 P}\right). \quad (48)$$

A power series expansion of the 2nd exponential gives

$$H'(u) = e^{iua}$$
$$\times\left(1 - \frac{(iua)^2}{\ln P} + \frac{(iua)^3}{\ln^2 P}\frac{1+\beta+\beta^2}{1+2\beta+\beta^2} + O\left(u^4a^4\right)\right). \quad (49)$$

As expected, the transfer function obtained for the single absorption line arrangement (Eq. (30)) is retrieved by putting $\ln P = Z$ and $\beta = 0$ in equation (48). At the 2nd order, the distortion does not depend on β and the relation between P, a and ε established for (Eq. (31)) is valid whatever β is. This means that, in the low distortion limit, the gain-zero arrangement $(0 < \beta < 1)$ is as efficient as the single absorption-line arrangement $(\beta = 0)$. The 3rd order contribution to the distortion is even slightly lower when $\beta \neq 0$ (see Eq. (49)) and numerical simulations show that the exact distortions obtained with both arrangements for the same values of P and a are very close in all the cases of practical interest. For $P = 190$, $a = 0.42$ and $\beta = 0.2$, we find in particular $D_\infty = 7.34\%$ instead of 7.30% with the single absorption-line arrangement.

5.2 Absorption-doublet arrangement

In all the systems considered up to now, the distortion of the pulse-envelope mainly result from the 2nd order term when it keeps small. One may reasonably expect to attain a better efficiency when this term cancels. This occurs in the gain-doublet arrangement for $\Delta = 1/\sqrt{3}$ (see Eq. (40)) but the output-pulse is then delayed with respect to the incident one ($A = -3G/4\gamma$, see Eq. (35)). Conversely there is advancement in the absorption-doublet arrangement $(G = -Z)$. The gain-factor $F(\Omega)$ of the medium then displays a unique flat minimum at $\Omega = 0$ $(F(0) = -3Z/2)$ and is maximum at infinity $(F(\pm\infty) = 0)$. The peak gain of the system is $P = \exp(3Z/2)$ and the transfer function of the system (see Eqs. (37–39)) may be written

$$H'(u) = e^{iua}\exp\left(-\frac{3(iua)^3/\ln^2 P}{1 + 3iua/\ln P + 3(iua)^2/\ln^2 P}\right). \quad (50)$$

At the lowest order of distortion, the transfer function and the output signal become

$$H'(u) = e^{iua}\left(1 - \frac{3(iua)^3}{\ln^2 P} + O\left(u^4a^4\right)\right) \quad (51)$$

$$E'(L,\theta) \approx E(0,\theta+a) - \frac{3a^3}{\ln^2 P}\frac{d^3E(0,\theta+a)}{d\theta^3}\cdot \quad (52)$$

When the incident pulse is bell-shaped, the (odd) distortion term originates a reduction of the pulse-advancement, a shortening of the pulse rise and a lengthening of its fall. These effects obviously keep small in the low distortion limit. In this limit D_∞ reads

$$D_\infty \approx D_\infty^{(3)} = \frac{3a^3}{\ln^2 P}\left\|\frac{d^3E(0,\theta)}{d\theta^3}\right\|_\infty = \frac{11.7a^3}{\ln^2 P} \quad (53)$$

where the last expression is obtained in the case of Gaussian pulses. The approximate distortion given by equation (53) is in surprising good agreement with the exact distortion. Fixing $D_\infty^{(3)} = 7.3\%$, we indeed verified that the exact distortion keeps in the range 6.6–8.2% when P (resp. a) varies from 10 to 10^5 (resp. from 0.32 to 0.94). As expected, the present arrangement is more efficient than those previously considered. The advancement $a = 0.42$ with a distortion $D_\infty = 7.3\%$ is in particular attained for $P \approx 30$ instead of 190 in the single absorption-line arrangement and 1.2×10^4 in the gain-doublet arrangement.

An other advantage of the absorption doublet arrangement is that it keeps relatively simple and open to an experimental realisation. Depending on the experiment, the parameter Δ can be brought to the value $1/\sqrt{3}$ by adjusting either the line splitting $(\omega_2 - \omega_1)$ or the line broadening γ. The peak-gain P and the advancement a being given, the pulse duration τ_p is determined by the value of γ $(\tau_p = \ln P/2\gamma a)$ and is about $4/\gamma$ for the previous values of P and a (resp. 30 and 0.42). This duration should obviously be long enough in order to facilitate the shaping of the incident pulse and the real-time observation of the output signal. Favourable time-scales (say $\tau_p > 100$ ns) can in particular be obtained at millimetre and optical wavelengths. The microwave experiment could be achieved on a gas of methyl fluoride with a device similar to that described in [5] and in Section 3. The rotation transitions $|J=1,K=0\rangle \to |J=2,K=0\rangle$ and $|J=1,|K|=1\rangle \to |J=2,|K|=1\rangle$ of this symmetric top molecule lie at a wavelength $\lambda \approx 3$ mm and their frequencies differ by about 1.1×10^7 Rad/s owing to the centrifugal distortion [25]. Since the relaxation is mainly determined by the intermolecular collisions, Δ is simply brought to $1/\sqrt{3}$ by adjusting the gas pressure in order that $\gamma = \frac{\sqrt{3}}{2}(\omega_2 - \omega_1) \approx 10^7$ Rad/s. With the previous parameters, this leads to a pulse-duration $\tau_p \approx 400$ ns. In the optical domain, comparable time scales would be attained in a cloud of cold atoms [26]. We suggest to use ^{39}K potassium atoms and to retain the line $\left|^2S_{1/2}, F=1\right\rangle \to \left|^2P_{3/2}, F=1\right\rangle$, easily split by a magnetic field. The linewidth γ is fixed by the lifetime τ_e of the excited state ($\gamma = 1/2\tau_e$ with $\tau_e \approx 26$ ns) and this is the line splitting which is now adjusted by means of the magnetic field in order to bring Δ to $1/\sqrt{3}$. For $P = 30$ and $a = 0.42$, this leads to a pulse duration $\tau_p \approx 4/\gamma \approx 8\tau_e \approx 200$ ns. Experiments are also feasible on a low-pressure gas in the infrared domain ($\lambda \approx 10$ μm) where numerous molecules present suitable rotation-vibration lines. The pulse-duration, then fixed by the Doppler broadening, would be one order of magnitude shorter than in the above proposals.

5.3 Extension to several doublets

We now consider the more general situation where r gain or absorption doublets, differing only by their gain G_k and their line-splitting parameter Δ_k, are involved in the dispersion mechanism. Each additional doublet provide two new degrees of freedom and the use of r doublets allows us in principle to cancel the distortion up to the order $2r$, the first term different from zero being that of order $n = 2r+1$. This way of search of the "ideal medium" is somewhat similar to the Butterworth procedure used in the design of analog filter [27]. Note however that the amplitude and the phase of the transfer function are taken into account in our approach whereas only the amplitude transmission is optimised in the Butterworth procedure.

By introducing the parameter $s = 1/\gamma\tau_p$ (further related to the relative pulse-advancement a) and the angles $\Psi_k = \mathrm{Arg}(1+\mathrm{i}\Delta_k)$, the complex gain-factor of the medium may be expanded under the form

$$\Gamma(u) = \sum_{k=1}^{r} \left(\frac{G_k}{1+\mathrm{i}\Delta_k + \mathrm{i}su} + \frac{G_k}{1-\mathrm{i}\Delta_k + \mathrm{i}su} \right) = \sum_{n=0}^{\infty} C_n \,(\mathrm{i}su)^n \tag{54}$$

where

$$C_n = 2\,(-1)^n \sum_{k=1}^{r} G_k \,(\cos\Psi_k)^{n+1} \cos\left[(n+1)\,\Psi_k\right]. \tag{55}$$

C_0 and sC_1 are respectively equal to the real gain-factor of the medium at the working frequency ($u = 0$) and to the relative pulse-advancement a. The distortion originates from the terms of index $n \geq 2$. By imposing that the distortion is zero up to the order $2r$, *i.e.* that $C_n = 0$ for every n ranging from 2 to $2r$, one obtains a system of $(2r-1)$ equations, homogeneous and linear with respect to the G_k. Taking into account that $0 \leq \Psi_k \leq \pi/2$ (Δ_k positive and finite), the solution to this system is

$$\Psi_k = \frac{2k-1}{2r+1}\frac{\pi}{2} \tag{56}$$

$$G_1 \cos^2\Psi_1 = G_2 \cos^2\Psi_2 = \cdots = G_r \cos^2\Psi_r = G_0 \tag{57}$$

where G_0 is a parameter independent of k. Since $\cos^2\Psi_r = 1/\left(1+\Delta_k^2\right)$, equation (57) shows that all the doublets equally contribute to the gain at the working frequency ($u = 0$). The pulse-advancement is easily derived from equations (55–57). One gets

$$a = sC_1 = -2sG_0 \sum_{k=1}^{r} \cos(2\Psi_k) = -sG_0. \tag{58}$$

There is advancement only if G_0 is negative, that is if all the doublets are absorption-doublets. In this case $F_{\max} = F(\pm\infty) = 0$ and the peak-gain P of the system is such that

$$\ln P = -F(0) = -C_0 = -2\sum_{k=1}^{r} G_k \cos^2\Psi_k = -2rG_0. \tag{59}$$

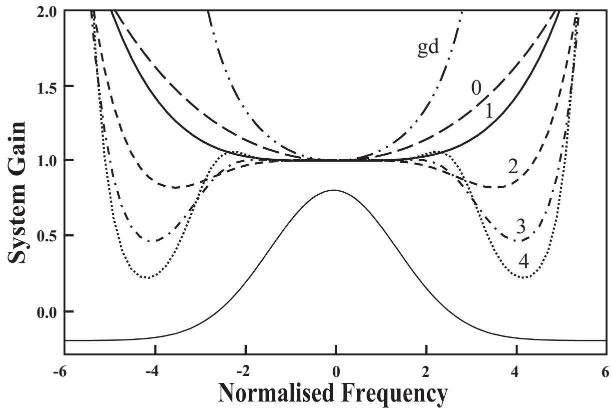


Fig. 9. Profiles of the system gain $|H'(u)|$ for different arrangements but the same values of the peak-gain and of the relative pulse-advancements ($P = 190$ and $a = 0.42$). The curves gd, 0, 1, 2, 3 and 4 respectively correspond to the arrangements involving a gain doublet ($\Delta = 3.5$), a single absorption-line, one, two, three and four optimised absorption-doublets. The normalised spectrum $\exp(-u^2/4)$ of the incident pulse (assumed to be Gaussian) is given for reference (lower isolated curve).

The parameters G_0 and s are simply related to P and a ($G_0 = -\ln P/2r$, $s = -a/G_0 = 2ra/\ln P$) and the transfer function of the system may finally be expressed under a general form involving the number r of doublets, the peak-gain P of the system and the relative advancement a

$$H'(u) = P \exp\left\{ -\frac{\ln P}{2r} \sum_{k=1}^{r} \left(\frac{1+\Delta_k^2}{1+\mathrm{i}\Delta_k + \mathrm{i}ua\,(2r/\ln P)} + \frac{1+\Delta_k^2}{1-\mathrm{i}\Delta_k + \mathrm{i}ua\,(2r/\ln P)} \right) \right\} \tag{60}$$

where

$$\Delta_k = \tan\Psi_k = \tan\left(\frac{2k-1}{2r+1}\frac{\pi}{2}\right). \tag{61}$$

Figure 9 shows the gain profiles $|H'(u)|$ obtained for the same values of P and a with 1, 2, 3 or 4 optimised absorption doublets. The gain-profiles corresponding to the single absorption-line and gain-doublet arrangements are given for comparison. As expected, the gain-profile in the vicinity of the working frequency is flatter (resp. much flatter) with the absorption-doublet ($r = 1$) than with a single absorption-line (resp. a gain-doublet). However, the improvement obtained by increasing the number of absorption-doublets, if it exists, is not well marked. This is not really a surprise since a too flat transmission curve may lead to a reduction of the 1st order dispersion, responsible of the pulse-advancement, and we are comparing the different systems for a same value of this advancement. A more precise comparison of the different arrangements obviously requires a direct examination of the distortion. In the case of r optimised absorption-doublets, the lowest order contribution to the distortion arises from the term of degree $(2r+1)$ in u and, at this order of approximation, the transfer function of the system and the output signal may be written under forms analogous to those given by equations (19, 20) in the general analysis

$$H'(u) \approx e^{iua}\left(1 + D_{2r+1}\,(iua)^{2r+1}\right) \quad (62)$$

$$E'(L,\theta) \approx E(0,\theta + a) + D_{2r+1}a^{2r+1}\frac{d^{2r+1}}{d\theta^{2r+1}}\left[E(0,\theta + a)\right] \quad (63)$$

with

$$D_{2r+1} = \left(\frac{s}{a}\right)^{2r+1} C_{2r+1} = \left(\frac{2r}{\ln P}\right)^{2r}\sum_{k=1}^{r} 2\cos^{2r}\Psi_k \cos\left[(2r+2)\,\Psi_k\right]. \quad (64)$$

At the lowest order, the uniform norm distortion reads

$$D_\infty^{(2r+1)} = D_{2r+1}a^{2r+1}\left\|\frac{d^{2r+1}}{d\theta^{2r+1}}E(0,\theta)\right\|_\infty = \left(\frac{\mu_r}{\ln P}\right)^{2r} a^{2r+1} \quad (65)$$

where μ_r is a numerical coefficient depending on the number r of doublets and on the shape of the envelope $E(0,\theta)$ of the incident pulse. For Gaussian pulses, we get $\mu_1 = 3.42$ (in agreement with Eq. (53)), $\mu_2 = 7.15$, $\mu_3 = 11.3$ and $\mu_4 = 15.8$. The general expression of the coefficients C_n (see Eq. (55)) allows us to determine the contributions to the distortion of order $n > 2r+1$. As one might fear, some of these contributions are far from being negligible even when $D_\infty^{(2r+1)}$ is small. For the largest values of r, they may even dramatically exceed that of order $(2r+1)$. In these conditions the uniform norm distortion derived from this only term (Eq. (65)) is not representative of the exact distortion. The case of a single absorption-doublet constitutes a noticeable exception. The distortions of order 3 and 4 clearly prevail over those of higher order. The 4th order distortion is only slightly smaller than the 3rd order one but it occurs in a different region of the pulse-envelope. This explains why $D_\infty^{(3)}$ actually provides a good approximation of D_∞, as noted in the previous subsection.

The above considerations naturally led us to proceed to numerical calculations of the distortion. More exactly, the relative advancement a being given, we determined for each arrangement the system peak-gain P for which D_∞ attains an *a priori* fixed level of distortion. Figure 10 displays the curves $P(a)$ obtained for $D_\infty = 7.3\%$ and a ranging from 0.1 to 1. In agreement with the classification of the different arrangements previously made on the basis of the gain-profiles (Fig. 9), the curves of Figure 10 confirm that the arrangements involving 1, 2 and 4 absorption doublets have nearly equal efficiencies (in the sense given to this word in the beginning of this section). The curve corresponding to 3 absorption-doublets (not shown for sake of clarity) lies between those obtained for $r = 2$ and $r = 4$. For the level of distortion considered,

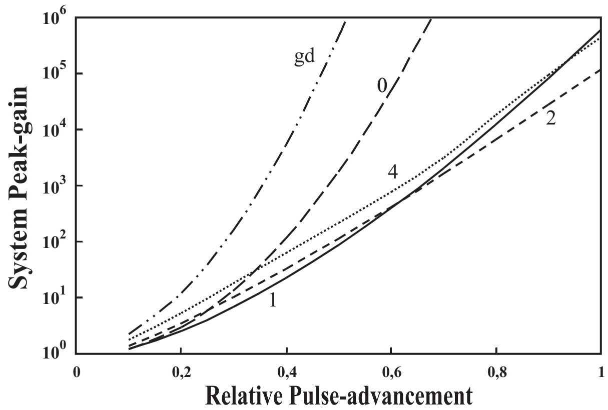


Fig. 10. System peak-gain P (logarithmic scale) as a function of the relative pulse-advancement a for $D_\infty = 7.3\%$. The meaning of the symbols is as in Figure 9.

the arrangement involving one absorption doublet (full line) is the most efficient up to $a = 0.6$ and this advancement is obtained for an amplitude peak-gain $P \approx 400$. Very large values of P are excluded in real experiments owing to the already mentioned problems of noise, instability and spurious signals whose importance increases in proportion to the gain. Imperceptible defects in the envelope of the incident pulse may in particular generate transients (wiggles) obscuring the expected output pulse for very large gains. Wiggles of moderate amplitude have been observed for $P \approx 190$ (see Fig. 3) despite of the care taken in the pulse shaping [5]. Even if further improvements in the pulse shaping are possible, this phenomenon limits P to a value that one may optimistically estimate to 10^4 (80 dB). This fixes the upper bound of the relative advancement a to about 0.8. For $0.6 < a < 0.8$, the single absorption-doublet arrangement is very slightly less efficient than its 2-doublets counterpart whose implementation is much more complex. For a practical application, the single absorption-doublet scheme may thus be considered as the best one in the whole domain $0.1 < a < 0.8$. As indicated before, equation (53) works in this range and allows us to analyse the effects of the different parameters. It shows that the advancement increases very slowly with the peak-gain (as $(\ln P)^{2/3}$) and is not very sensitive to the level of tolerated distortion ($a \propto D_\infty^{1/3}$) and to the exact profile of the incident pulse (provided that it is ideally smooth!). For example, an increase of the peak-gain from 10^4 to the unrealistic value 10^5 would result in an increase of the advancement by only 16%. Conversely the advancement is reduced by only 21% if the tolerated distortion is reduced by half. For a same distortion, it decreases by only 17% if the Gaussian pulse is replaced by a sech-pulse of same duration τ_p.

6 Summary and discussion

In this work we have studied in what conditions electromagnetic pulses can propagate at a negative group velocity without significant change in their characteristics.

The required anomalous dispersion is associated to narrow absorption and/or gain lines of the medium. A broad-band amplifier or absorber should eventually complete the latter in order to compensate the absorption or the amplification of the medium at the working frequency. The transfer function of the global system is written in a fairly general case and some important properties are derived from it. The pulse-advancement is in particular related to the spectral profile of the gain-factor and is shown to be maximum when the working frequency corresponds to a well-marked minimum of gain. The 1st order distortion (mainly a frequency change) then cancels and the advancement obtained at this order of approximation coincides with that of the centre-of-gravity of the pulse-envelope whatever the distortion is. The higher order distortion is examined in this case and a special attention is paid to the 2nd order contribution which prevails in most low-distortion systems and whose main effect is a narrowing of the pulse amplitude-profile by a factor close to unity and its magnification by the same factor (area conservation). Explicit expressions of the transfer function and of the distortion are given for the experiments involving a single absorption-line or a gain doublet. The uniform norm distortion D_∞ is expressed as a function of the relative pulse-advancement a (ratio of the advancement A over the pulse width τ_p)[2] and to the system peak-gain P (which is nothing else that the ratio of the peak value of the medium gain over its value at the working frequency). The dimensionless character of all our expressions allows us to simply analyse experiments made in quite different domains. Surprisingly enough, the direct experimental evidences of significant pulse-advancements with low distortion are very few and the most convincing keeps that we achieved in 1985 with a single absorption-line arrangement [5]. Insofar as our work has been overlooked in most subsequent publications on the subject with few exceptions [28,29], it is worth recalling that we obtained $a = 0.42$ with $D_\infty < 7.3\%$. Conversely the pulse advancement recently attained by Dogariu *et al.* [7,8] by using a gain-doublet arrangement is modest ($a = 0.03$) and, paradoxically enough, their results could be reproduced in the single absorption-line arrangement with a better transmittance of the medium. The present limits of the gain-doublet experiments originate in that of the actually usable gain. The required gain-profile is indeed created by a bichromatic Raman pumping and four-wave mixing processes originate a splitting of the transmitted probe pulse (*i.e.* a considerable pulse-reshaping) as soon as the line gain-factor G exceeds one [9]. We suggest overcoming this difficulty by using a monochromatic instead of bichromatic Raman pumping and a level-scheme with a doublet of final states.

As stated a long time ago by Garrett and McCumber [3] and recalled in our introduction, the propagation of light pulses at a negative velocity is not at odd with the causality principle and the special relativity theory. The phenomenon occurs only with ideally smooth pulses and it is irrelevant to make any correspondence between homologous points of the envelopes of the pulses entering and leaving the medium. One may remark that Mother Nature strongly resists to a breaking of its principles even when this breaking is only apparent, as it is the case here. The observation of significant pulse-advancements with low distortion indeed involves systems whose gain, equal to one at the working frequency, takes dramatically large values at other frequencies. For example, the values $a = 0.42$ and $D_\infty = 7.3\%$ are attained for $P = 190$ and $P = 1.2 \times 10^4$, respectively with the single absorption-line and gain-doublet arrangements. When the peak-gain P is very large, the slightest defects in the envelope of the incident pulse result in important parasitic signals at the system output. Furthermore amplified spontaneous emission, instabilities or undesired laser effects may completely obscure the transmitted pulse in optical systems (independently of the above-mentioned problems associated to a bichromatic pumping). One may thus consider that the most efficient system is that having the smallest peak gain for imposed values of a and D_∞. From this viewpoint, the single absorption-line arrangement is much more efficient than the gain-doublet scheme. A comparable efficiency is obtained with the arrangement involving a narrow dip in a gain profile with the advantage of a perfect transparency of the medium at the working frequency whatever the advancement is. A better efficiency is expected when the 2nd order distortion cancels. The simplest arrangement complying with this condition consists of a doublet of absorption-lines. The values $a = 0.42$ and $D_\infty = 7.3\%$ are then attained for a peak gain $P = 30$, actually much smaller than with the previous arrangements. The distortion can be cancelled up to the order $2r$ by using r absorption doublets. Disappointingly enough, this does not lead to a significant increase of the efficiency. The (single) absorption-doublet arrangement is even the most efficient for $P < 400$. It is fairly simple and we propose different schemes for its experimental implementation.

The limit imposed to the system peak-gain P in order to prevent the pollution of the output pulse by undesired signals determine the largest advancement which can be attained for a given level of distortion. By very carefully shaping the incident pulse and working at long enough wavelength (in order to reduce the spontaneous emission), one may optimistically expect to use systems with a peak gain $P = 10^4$. Taking again $D_\infty = 7.3\%$ as tolerated level of distortion, we get $a \approx 0.8$ with the absorption-doublet arrangement. This value may be considered as an upper limit of the relative advancement actually attainable in a concrete experiment with nearly Gaussian pulses. Note that relative advancements a of the order of 0.5 can be attained for reasonable peak-gains ($P < 100$) but that a further increase of the advancement is paid at a high price in terms of gain.

Two general comments may be made about the propagation of light pulses at a negative group velocity. We first remark that the pulse-distortion mainly (or for an important part) originates in the variations of the medium transmittance as a function of the frequency. The pulse

[2] Let us recall that in our notation τ_p is the half-width at 1/e of the amplitude profile of the incident pulse.

shapes and advancements inferred from measurements of the only dispersion-characteristics are thus quite disputable. Second, the distortion does not depend on the absolute value A of the pulse-advancement but only on its relative value a. Since the distortion cancels for $a \rightarrow 0$, it is not a problem to conciliate low distortion with pulse-advancements A large with respect to the luminal transit time L/c. This is simply achieved by using lines of width γ very small compared to αc or gc (as it is the case in all the experiments discussed here) and long enough pulses. The true challenge is obviously to attain advancements comparable to the pulse-width.

The formalism developed in this paper is immediately applicable to the study of ultraslow propagation of light pulses in resonantly dispersive media. See the recent review by Boyd and Gauthier [29]. It indeed suffices to change the absorption or gain coefficients of the lines appearing in our expressions into their opposites to transform advancements into delays. It is quite instructive to compare the results obtained in each case for a same time-shift of the pulse in modulus. We restrict the comparison to the case of a single gain or absorption line. As a matter of fact, the experiments of ultraslow propagation have been achieved with an arrangement involving a dip of sub-natural width in a wide absorption-profile (atomic line). This arrangement is the inverse of the gain-zero arrangement considered in the present paper and is equivalent to a single gain-line arrangement owing to the narrowness of the dip. The main difference between the experiments at slow and negative group velocity lies in the system peak-gain P required in order to attain a significant time-shift (relative pulse-delay or advancement). P is equal or close to 1 in the first case whereas it takes large values originating the above-discussed experimental difficulties in the second case. In the low distortion limit, the output signal is described by the same equation (Eq. (25)) but the distortion parameter ε takes opposite values. The narrowing and the magnification of the amplitude-profile of the pulse obtained at negative group velocity now become a broadening and attenuation by the same factor. This analytical result is well reproduced by numerical simulations using the transfer function associated to a single absorption-line (Eq. (30)), Z being changed into $-Z$ and a into $-a$. Such numerical simulations allow us to explore the phenomena occurring when the 2nd order approximation is no more valid. For increasing values of $|\varepsilon|$, the distortion increases much more slowly when the group velocity is subluminal rather than negative. This is not surprising and, roughly speaking, results from the alternated structure of the power series expansion of the transfer function when the propagation is subluminal. A calculation made with the parameters of the experiment of Hau *et al.* [30] shows that despite the large modulus of the distortion parameter ($\varepsilon \approx -0.7$), the output pulse keeps nearly Gaussian. It is about twice as wide as the incident pulse, its peak intensity is about four times smaller and the relative delay is very close to the value derived from the 1st order approximation ($a = -3.3$). These theoretical results are in satisfactory agreement with the experimental ones (see in particular Fig. 3 in [30]). Conversely, the same calculation in the case of a negative group velocity ($\varepsilon = +0.7$) shows that the output pulse is completely distorted with an oscillatory structure and a peak-intensity fourteen larger than that of the incident pulse. It is obviously meaningless to speak of pulse-advancement in such conditions.

Finally the methods used in our study of the propagation of electromagnetic pulses at a negative group velocity can easily be adapted to that of the (apparent) superluminal transmission of pulses through one-dimensional systems with photonic gap. See the recent reviews by Chiao and Steinberg [28] and by Nimtz and Heitman [31]. This problem is a subject *per se* and will be examined in a forthcoming paper. However some preliminary remarks may be already made. First the waves involved in these systems are not travelling waves and important phenomena occur at the boundaries. Strictly speaking, group velocity is not a relevant concept and it is preferable to speak only of group delays. Second the group delays are actually smaller than the luminal transmission time but are never negative[3] as in the systems considered in the present paper. Third the systems involving a single barrier (*e.g.* a microwave waveguide with one part under cut-off) have no minimums of transmission. According to our general analysis, the pulse-distortion then appears as early as at the 1st order of approximation and these systems are not suitable for the purpose of attaining large pulse-advancements[4] with low distortion. On the contrary, such minimums of transmission exist in the systems involving a stack of dielectric layers [28,31,32], periodic fibre Bragg gratings [33] and double-barrier photonic band-gaps [34,35]. The properties of these systems are analogous to those of the arrangement involving a single absorption-line, with a 2nd order distortion. Significant relative advancements have been attained in recent experiments involving direct observations in the time-domain [32,33,35]. Suitable modifications of these experiments would permit to cancel the 2nd order distortion as with the absorption-doublet arrangement considered in the present paper.

We thank D. Gauthier and M.Stenner for providing us details about their experiments on a bichromatically driven Raman-amplifier, W. Mecklenbraueker for useful indications on the linear systems theory and O. Ramaré for his help in some mathematical demonstrations. One of us (BM) would like to thank J.C. Garreau for fruitful discussions about the optical pumping and the Raman processes in atomic physics.

Note added in proof

The experiment involving electromagnetically induced absorption, proposed at the end of Section 3, has been successfully achieved by Akulshin *et al.* [36]. In our notations,

[3] As far as we know, this is true for all the experiments achieved up to now but we are not aware of a general demonstration of this conjecture.

[4] With respect to a pulse that would be transmitted at the luminal velocity.

the relative advancement and the uniform norm distortion obtained in this experiment are respectively 10% and 3%. Besides, negative group delays have been very recently evidenced in the transmission of radiofrequency pulses through a 1D photonic band gap structure [37]. The negative delays are attributed to significant losses in the system. See also [38].

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